Interval α-Discounting Method for MDCM

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Abstract.

This paper is an extension of our previous work on α-Discounting Method for MCDM ([1], [2], [3]) from crisp numbers to intervals.

Keywords:
Multi-Criteria Decision Making (MCDM), Analytical Hierarchy Process (AHP), α-Discounting Method, Fairness Principle, parameterize, pairwise comparison, n-wise comparison, consistent MCDM problem, inconsistent MCDM problem

1. Introduction.

In 2010 we have introduced a new method [3], called α-Discounting Method for Multi Criteria Decision Making, which is an alternative but also a generalization of Saaty's Analytical Hierarchy Process (AHP). α-Discounting Method works for any n-pairwise comparisons, $n \geq 2$, that may be linear or non-linear, or may be equations or inequalities. It transforms all preferences into a system of equations and/or of inequalities, that is later solved algebraically.

Since *Saaty's AHP* is not the topic of this paper, we'll not present over here. Neither our α-*Discounting Method for Multi Criteria Decision Making* is recalled. The interested reader may get them in [7], and respectively [1], [2], and [3].

2. A consistent example

Let have the set of criteria be $C = \{C_1, C_2, C_3\}$, and the set of preferences P be:

a. C_1 is twice or three times as important as C_2;

b. C_2 is one or one and half times as important as C_3.

Solution.

Let x represents the value of C, y of C_2, and z of C_3. $x > 0, y > 0, z > 0$.

We form the algebraic interval system:

$$\begin{cases} x = [2,3]y \\ y = [1,1.5]z, \end{cases}$$

where $[2,3]$ and $[1,1.5]$ are intervals.

Replacing the second equation into the first, one gets:

$$x = [2,3]y = [2,3] \cdot [1,1.5]z = [2 \cdot 1, 3 \cdot 1.5]z = [2,4.5]z.$$

The general solution of this system is:

$$\langle [2,4.5]z, [1,1.5]z, z \rangle \text{ where } z > 0.$$

We divide this vector components by z, and we get:

$$\langle \underset{C_1}{[2,4.5]}, \underset{C_2}{[1,1.5]}, \underset{C_3}{1} \rangle.$$

We don't know exactly what to mean by normalization when dealing with intervals, but it is clear that $C_1 > C_2 > C_3$.

3. A second consistent example

Criteria: $C = \{C_1, C_2, C_3\}$, and the set of preferences P, same as in the previous example, but adding one more:

a. C_1 is twice or three times as important as C_2;
b. C_2 is one or one and half times as important as C_3;
c. C_3 is $\frac{1}{4}$ or $\frac{1}{2}$ times as important as C_1 .

Solution.

With same notations x, z and z representing the values of C_1, C_2 and respectively C_3, we form the algebraic system:

$$\begin{cases} x = [2,3] \cdot y \\ y = [1,1.5] \cdot z \\ z = [0.25, 0.50] \cdot x, \end{cases}$$

with $x > 0, y > 0, z > 0$.

The determinant of the system is:

$$\begin{vmatrix} 1 & -[2,3] & 0 \\ 0 & 1 & -[1,1.5] \\ -[0.25,0.50] & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & [-3,-2] & 0 \\ 0 & 1 & [-1{,}5,-1] \\ [-0.50,-0.25] & 0 & 1 \end{vmatrix}$$
$$= 1 — [-3,-2] \cdot [-0.50,-0.25]$$
$$= [1,1] +][(-3) \cdot (-1.59) \cdot (-0.5), (2) \cdot (-1) \cdot (-0.25)]$$
$$= [1,1] + [-2.25,-0.50] = [1-2.25, 1-0.50] = [-1.25, 0.50]$$
$$\neq [0,0].$$

Let's parameterize the system, using $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$ in order to discount each interval coefficient. We get:

$$\begin{cases} x = \alpha_1 [2,3] y \\ y = \alpha_2 [1,1.5] z \\ z = \alpha_3 [0.25, 0.50] x. \end{cases}$$

The determinant of the parameterized system is:

$$\begin{vmatrix} 1 & -\alpha_1[2,3] & 0 \\ 0 & 1 & -\alpha_2[1,1.5] \\ -\alpha_3[0.25,0.50] & 0 & 1 \end{vmatrix} = 1 - \alpha_1\alpha_2\alpha_3[2,3][1,1.5][0.25,0.50]$$
$$= [1,1] - \alpha_1\alpha_2\alpha_3[0.50,2.25] = [0{,}0].$$

Whence $\alpha_1\alpha_2\alpha_3[0.50, 2.25] = [1,1]$, hence $\alpha_1\alpha_2\alpha_3 = \left[\frac{1}{2.25}, \frac{1}{0.50}\right] = \left[\frac{4}{9}, 2\right]$.

For equitable discount, let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$. Then

$\alpha^3 = \left[\frac{4}{9}, 2\right]$, whence $\alpha = \left[\sqrt[3]{4/9}, \sqrt[3]{2}\right] \simeq [0,76, 1.26]$.

There, the system is altered with the same proportion $\alpha = [0,76, 1.26]$ each equation, and it becomes:

$$\begin{cases} x = [2,3] \cdot \alpha \cdot y = [2,3] \cdot [0.76, 1.26]y = [1.52, 3.78]y \\ y = [1, 1.5] \cdot \alpha \cdot z = [1, 1.5] \cdot [0.76, 1.26]z = [0.76, 1.89]z \\ z = [0.25, 0.50] \cdot \alpha \cdot x = [0.25, 0.50] \cdot [0.76, 1.26]x = [0.19, 0.63]x \end{cases}$$

From the first two equations we get:

$$\begin{cases} x = [1.52, 3.78]y = [1.52, 3.78] \cdot [0.76, 1.89]z = [1.16, 7.14]z \\ y = [0.76, 1.89]z \end{cases}$$

The third equation

$$1 \cdot z = [0.19, 0.63]x$$

is equivalent to

$$x = \frac{1}{[0.19, 0.63]}z = \left[\frac{1}{0.63}, \frac{1}{0.19}\right]z = [1.59, 5.26]z.$$

Therefore we got the following approximation that we can call reconciliation of the first equations together, that give us:

$$x = [1.16, 7.14]z$$

with respect to the third equation that gives us:

$$x = [1.59, 5.26]z.$$

We see that the intervals $[1.16, 7.14]$ and $[1.59, 5.26]$ are close to each other.

The solution vector of the parameterized system, for $\alpha = [0.76, 1.26]$ is

$$\langle [1.16, 7.14]z \text{ or} [1.59, 5.26]z, [0.76, 1.89]z, z \rangle.$$

We divide by $z > 0$ and we get:

$$\langle \underset{C_1}{[1.16, 7.14] \text{ or } [1.59, 5.26]}, \underset{C_2}{[0.76, 1.89]}, \underset{C_3}{1} \rangle.$$

It's not necessary to normalize. We can see that:

$$C_1 > C_2 \text{ and } C_1 > C_{3.}$$

To compare C_2 with C_3, we see that in general

$$C_2 > C_3,$$

since the interval $[0.76, 1.89]$ has a bigger part which is

$$(1, 1.89] > 1 \text{ when } C_2 > C_3,$$

and a smaller part $[0.76, 1)$ when $C_2 > C_3$,

while a single case $[1, 1] = 1$ when $C_2 = C_3$.

4. Inconsistent example

Same criteria and the first two preferences. Only the third preference is changed as in the below third equation.

$$\begin{cases} x = [2, 3]y \\ y = [1, 1.5]z \\ z = [3, 3.5]x. \end{cases}$$

From first and second equations, we get

$$x = [2, 3]y = [2, 3], [1, 1.5]z = [2, 4.5]z.$$

From the last equation:

$$x = \frac{1}{[3, 3.5]} = \left[\frac{1}{3.5}, \frac{1}{3}\right] \simeq [0.29, 0.33]z$$

which is different from $[2, 4.5]z$.

Parameterized in the same way as before:

$$\begin{cases} x = \alpha_1[2, 3]y \\ y = \alpha_2[1, 1.5]z \\ z = \alpha_3[3, 3.5]x. \end{cases}$$

We similarly get from the first two equations:

$$x = c[2, 4.5]z$$

and from the last equation:

$$x = \frac{1}{[3, 3.5]} \cdot \frac{1}{\alpha_3} \simeq \frac{1}{\alpha_3}[0.29, 0.33]z.$$

Whence:

$$\alpha_1\alpha_2[2, 4.5]z = \frac{1}{\alpha_3}[0.29, 0.33]z$$

or

$$\alpha_1\alpha_2\alpha_3[2, 4.5] = [0.29, 0.33],$$

hence

$$\alpha_1\alpha_2\alpha_3 = \frac{[0.29, 0.33]}{2, 4.5]} = \left[\frac{0.29}{4.5}, \frac{0.33}{2}\right] \simeq [0.064, 0.165].$$

Considering an equitable discount we set $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$.

Hence $\alpha_1\alpha_2\alpha_3 = [0.064, 0.165]$ becomes $\alpha^3 = [0.064, 0.165]$,

whence $\alpha = \left[\sqrt[3]{0.064}, \sqrt[3]{0.165}\right] \simeq [0.400, 0.548]$.

Whence we get:

$$x = \alpha_1\alpha_2[2, 4.5]z = [0.400, 0.548][0.400, 0.548][2, 4.5]z = [0.32, 1.55]z$$

or

$$x = \frac{1}{\alpha_3}[0.29, 0.33]z = \frac{1}{[0.400, 0.548]} \cdot [0.29, 0.33]z = \left[\frac{0.24}{0.548}, \frac{0.33}{0.400}\right]z$$
$$= [0.529, 0.825]z$$

and

$$y = \alpha_2[1, 1.5]z = [0.400, 0.548][1, 1.5]z = [0.400, 0.822]z.$$

The solution of the parameterized system is:

$$\langle [0.32, 1.55]z \text{ or } [0.529, 0.825]z, [0.400, 0.822]z, z \rangle.$$

We divide by $z > 0$ and we get:

$$\langle \underset{C_1}{[0.32, 1.55] \text{or } [0.529, 0.825]}, \underset{C_2}{[0.400, 0.822]}, \underset{C_3}{1} \rangle.$$

Clearly $C_2 < C_3$.

Then $C_1 < C_3$ for most part of its values, i.e.

for $[0.32, 1)$ or $[0.529, 0.825]$

and $C_1 > C_3$for $(1, 1.55)$

while $C_1 = C_3$ for $[1, 1]$.

To compare C_1 and C_2 it is more complicated.

Conclusion.

In this paper we have constructed two consistent examples and one inconsistent example of decision making problems, where the preferences use intervals instead of crisp numbers in comparisons of preferences. The results are, of course, more complicated.

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